SUPPLEMENTARY INFORMATION Coexistence of critical phenomena: The concept of manifold multi-spectral criticality

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This supplementary information material's first section aims to make reading the article's main text easier, especially for readers new to econophysics & and sociophysics areas of the science of complexity, by introducing several key paradigms. In the second section of this SI document, we provide an in-depth discussion of the algorithm used to obtain our results communicated in the main text of the manuscript. Furthermore, in the third section this document presents material expanding on the some of the most exciting observations and results of the main text which are signalled but remain outside the scope of the manuscript and will be further developed and communicated in the near future.

1. COMPANY MARKET PARADIGMS - KEY CONCEPTS INTRODUCED

First, we introduce the paradigms' general foundation and inspiration for our approach. Any model aspiring to describe the socio-economic reality should meet the paradigms describing companies' markets. They are crucial generic assumptions (stimulated by economic and social reality). Here is the "decalogue" of the companies' market competing in the presence of state interventionism.

(i) The competitive market is an evolving probabilistic/stochastic network. The market uncertainty accompanies it, generating an irremovable market risk. This risk lies in all kinds of forecasts.

(ii) The essential market feature is competition, i.e., the struggle of entities for domination. As a result, companies can form, merge, or go bankrupt. The following channels for the formation of companies are available: (a) the creation of daughter companies (spin-offs) by existing companies and (b) the formation of start-ups on the market. Let us add that there are few start-ups, and their market share is negligible. Therefore for simplicity, our model does not consider the start-up type of enterprises.

(iii) An essential feature of the interaction of entities with the market is feedback, which allows for the adaptation of entities to market conditions, their selection, and mutation. This type of coupling leads to a non-linear interactions – an essential feature of complex reality. We consider this reality a simplified analogon of evolving ecological systems with the principle of natural selection due to competition.

(iv) Innovation is understood as increasing the company's technological level. In the long term, it is one of the pillars of companies' survival in the market and development. Another pillar is the companies' market share (which we already discussed in our earlier work [36]).

(v) The companies compete on the market in the presence of a broadly understood state intervention. The state deals with indirect intervention in the market through legislation and through direct interaction through supporting companies and the purchase of market shares. The state acts as a legislator by establishing the rules for market functioning. In addition, it acts as an arbitrator/conciliator and a player in the market through state-owned companies. Hence, the state has the tools to beat any competitor. Therefore, for the free market not to monopolise while being liquidated, the state must be self-limiting.

(vi) The market reacts to interventionism so that the number of companies with a chance of surviving in the market decreases as the level of interventionism decreases. Moreover, there is a state interventionism threshold which is a critical one.

(vii) The principal condition for a competing company market assumes that the number of companies on the market, which cannot go bankrupt, is not lower than a certain minimum number of companies. However, this does not exclude the possibility of mergers of the companies. Which, in the case of a small population of companies on the market, is to be neglected. Therefore, this condition introduces partial protection against market monopolisation.

(viii) There exists a spectrum of partially stationary macrostates in the market (i.e., some non-equilibrium states in a statistical sense as discussed in the main text. In addition, the market may have a spectrum of unstable plateaus. However, in this work, we only deal with the partially stationary macrostates because they are common macrostates of the market of competing companies in the presence of state interventionism.

(ix) Phase transitions are common in the real world. For example, we can deal with continuous phase transitions, including critical phenomena. Therefore, models which do not exhibit any phase transitions do not fit the description of reality over a sufficiently long market duration.

(x) The market of competing companies is the foundation of all other markets (e.g., financial markets) – it is connected with them directly or indirectly.

2. HOW THE ALGORITHM WORKS: LOCAL STOCHASTIC DYNAMICS EXPLAINED IN DETAIL

The algorithm key points listed below represent the local stochastic dynamics defined for a single MCS. Note: All generated random numbers belong to the interval $\langle 0, 1 \langle$ and are drawn from a uniformly distributed random number generator. These numbers are, of course, pseudo-random numbers due to generator's limitations. Thus, the algorithm works as follows:

(1) It systematically picks a single company from $N(t)$ companies, i.e., the temporary location of the company.

(2) Next, it calculates the probability of the company's survival $p_i(t_{MCS})$ defined (for each MCS) by Eq. (24) in the main text, where index $i = 1, 2, \ldots, L^2$, numbers sites of the substrate square lattice.

(3) **Left bifurcation branch** of the 'binary tree of life' shown in Fig. 12 in the main text. It illustrates the non-equilibrium game of life in which the system under consideration takes part.

Namely, a random number r' is drawn. Because the intervention is successful here, i.e., $r' < q$, then the company's technology level, $A_i(t_{MCS})$, increases to its new value $A_i^{new}(t_{MCS}) = A_i(t_{MCS}) + \tilde{r}\eta\,[F(t-1) - A_i(t_{MCS})]$, where $\tilde{r}$ is another random number drawing, while the lattice site i is occupied by the company. We are forced to use two timescales here (see section 2 in this Supplementary Information below for details). In the main text, we used in Eq. (18) determination of coarse graining over time, $A_i(t) = A_i^{new}(t_{MCS})$, where time in t_{MCS} concerns $t-1$ MCS/site. We deal with an identical situation in item (7) in the analogous equation.

The technology level increases (according to the above equation) due to the purely external/exogenous technology diffusion driven by the technology level of the world leader/frontier, $F(t)$. However, the company copies the technology of the frontier imperfectly. Indeed, the random number $\tilde{r}$ is responsible for this imperfection, while probability $0 \leq \eta \leq 1$ defines the efficiency of state intervention in increasing the level of a company technology. Thus, η can be seen as a parameter that measures the quality of this interventionism.

Otherwise, when $r' \geq q$, the algorithm goes to the item (5).

(4) Still another random number r'' is drawn. If $r'' < \lambda$, then the algorithm goes to the item (6).

Otherwise, i.e., when $r'' \geq \lambda$, the company becomes passive in the current MCS, and the algorithm returns to the starting point, i.e., item (1), beginning the next MCS.

(5) **Right bifurcation branch** shown on the 'binary tree of life' presented in Fig. 12 in the main text.

The intervention is unsuccessful, i.e., the random number $r' \geq q$. Next, the random number r''' is drawn. If $r''' < p_i$, the algorithm goes directly to item (6).

Otherwise, i.e., when $r''' \geq p_i$, the company goes bankrupt (disappears from the lattice), and the algorithm returns to the starting point, i.e., to item 1, beginning the next MCS.

(6) The company attempts to move to one of the four randomly chosen neighbouring sites. If the chosen site is occupied, it does not move and stays in its place. However, as described below in item (8), it interacts with the company in that chosen occupied site in the current MCS.

(7) If the randomly selected lattice site, i, is empty, the company moves to this site and checks if this new site i has any nearest-neighbour site occupied by any other company. If there are no such neighbours, the company's technology level grows freely according to the formula $A_i^{new}(t_{MCS}) = A_i(t_{MCS}) + r[F(t-1) - A_i(t_{MCS})]$, where r is a subsequent random number drawn from the uniform distribution. This growth corresponds to purely external/exogenous technology diffusion. The company copies the technology of the world frontier imperfectly. Solely the random number r is responsible for this imperfection.

Otherwise, if the nearest neighbourhood of site i is occupied, a company from this neighbourhood is selected randomly for interaction. At the same MCS, if there is only one company in this neighbourhood, then the interaction occurs with that company.

(8) There are two kinds of interaction between companies leading to two mechanisms of inner/endogenous technology diffusion/spreading: (i) the merging of two companies or (ii) creating a new company or spin-off. So, as described in item (6), the company moves to a new (empty) i^{th} lattice site. If this company finds a randomly chosen company at the nearest-neighbour site j, then with probability b, the company at site i merges with the company at site j. The company at site j disappears from the lattice – it may be a hostile takeover of a competing company or simply the purchase of a business. The technology of the company at the i^{th} site changes according to the relation $A_i^{new}(t_{MCS}) = \max[A_i(t_{MCS}), A_j(t_{MCS})]$. The company at the i^{th} site also takes over the shares of the merged company, so $\omega_i^{new}(t_{MCS}) = \omega_i(t_{MCS}) + \omega_j(t_{MCS})$. It should be emphasised that the j site may be re-filled by some company in any other MCS within a given MCS/site.

Otherwise, with probability $1 - b$, companies at site i and its nearest-neighbour site j create a spin-off – a new company at previously empty lattice site k. Site k of the company is chosen randomly from the nearest and next-nearest neighbouring empty sites of the company located at site i. The spin-off appears only if its chosen site is empty. In this case, none of the companies disappear from the system. The spin-off's technology level is $A_k(t_{MCS}) = max[A_i(t_{MCS}), A_j(t_{MCS})]$ and the shares are $\omega_k(t_{MCS}) = \omega_s[\omega_i(t_{MCS}) + \omega_j(t_{MCS})]$, where fixed number $\omega_s \in \langle 0, 1 \langle$. Let us add that ω_s as a model parameter could also be a random number (and not just fixed permanently, as we take for simplicity). Because the normalisation condition of shares has to be met, the shares of the company located at i decrease by $\omega_s\omega_i(t_{MCS})$ and the company at site j by $\omega_s\omega_j(t_{MCS})$.

(9) The algorithm returns to item 1 until all the $N(t)$ companies have been chosen, ending the current MCS/site.

It is worth noting that the parameter η defining the quality of intervention can also be interpreted as a catalyst for a catalytic non-equilibrium chemical reaction of the following type: $A + F \xrightarrow{\eta} A^{new}$. Quantitatively, it is described by the equation given in item (3).

Two-time scales

In the MC algorithm, we consider, as usual, two-time scales: (i) Monte Carlo step (MCS) or a single draw (MC "second") and (ii) Monte Carlo step per lattice site (MCS/site or MC "minute"). At time scale (i), time t_{MCS} measures the total number of draws (seconds).

For time scale (ii), however, the situation is more complicated. The current number of companies on the lattice, $N(t_{MCS})$, depends on the time t_{MCS}. The point is that each MCS/site should count the same number of MC seconds, regardless of the current number of companies on the lattice. On the other hand, the point is not to randomise empty sites in the lattice, making the algorithm more practical. Therefore, a single MCS/site equals the number of lattice sites. However, it consists of two components: (i) the number of single draws (i.e., the number of MC seconds) equal to the number of companies at the beginning of this MCS/site and (ii) co-opted virtual seconds up to L^2 sites already without drawing. Indeed, thanks to this extension, the size of a single MCS/site is constantly equal to L^2 (regardless of the current number of companies). Thus, in a single MCS/site, each company has (on average) one chance of becoming active, which is as it should be.

We denote the current number of MCS/site as t. There is a simple relationship between t_{MCS} and t. Namely,

$$t_{MCS}(t) = \sum_{t'=0}^{t-1} N(t'),\ t = 1, 2, 3, \ldots, \tag{S1}$$

where t' numbers the next MCS/site. Thanks to the above relation, computer simulation can easily determine the total number of draws made up to t. From the above, there is a direct complementary relationship between the total number of virtual seconds $\tilde{t}_{MCS}(t)$ and $t_{MCS}(t)$ of the form,

$$\tilde{t}_{MCS}(t) = tL^2 - t_{MCS}(t). \tag{S2}$$

As one can see, both of the above expressions add upsides (as they should) to tL^2. Furthermore,

$$t'_{MCS}(t) = (t-1)L^2 + \Delta t_{MCS}(t), \tag{S3}$$

where $t'_{MCS}(t)$ is the integer number of MC seconds and $0 \leq \Delta t_{MCS}(t) \leq L^2$ is the current number of MC seconds (drawn and/or virtual) over a time interval $[t-1, t]$.

Thus, time in the system's dynamics can be equivalently expressed only in seconds or in Monte Carlo minutes and seconds.

3. GIANT COMPONENT EMERGENCE IN THE MIXED PERCOLATION-LIKE PHASE TRANSITION

In Fig. S1, we present two typical snap-shot pictures showing the lattice macrostates just before the critical threshold (left panel), i.e., for $z < z_c$ and just after the threshold (right panel), i.e., for $z > z_c$.

The indivisible giant component, containing two types of nested percolating clusters, is visible just after the threshold (right panel), i.e., for $z > z_c$. We

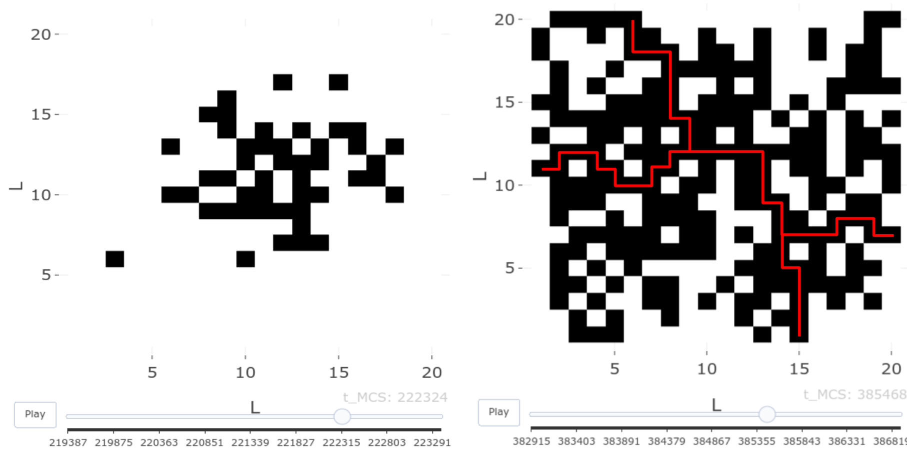

Fig. S1. Two snapshot pictures of agent deployments on a square lattice (of the linear size $L = 20$) in a partial stationary macrostate. Black squares locate lattice sites staffed by agents at once, while white areas are empty. More precisely, the lattice sites are defined by black and white square centres, see the right panel of Fig. S2. In the right panel, one can see many triads like the ones shown in the left panel in Fig. S2. In addition, a lot of linear triads (three agents lined up in a straight line) are visible here. The time range here covers the last 4 000 MCS (ending at 1 800 MCS/site – see Fig. 6 in the main text). Both panels concern macrostates lying on the green diagonal straight line in Fig. 4 in the main text (i.e., for $k = 0.5295676$). The left panel concerns the macrostate for $z = 0.481138 < z_c = 0.4937516$ (where $q = 0.870$ and $\lambda = 0.340$), while the right panel concerns the macrostate for $z = 0.506453 > z_c$ (where $q = 0.8877$ and $\lambda = 0.3581$). The number of agents in the lattice in the left panel is $N(t_{MCS} = 222\,324) = 46$, and in the right panel, $N(t_{MCS} = 385\,468) = 218$. As one can see, only for $z > z_c$ is it possible to observe the percolating path (the red broken multi-branched line going through the centres of black squares connected by edges) and hence a giant component/cluster. This component is immersed in a larger, complete cluster where common single/shared vertices connect the black squares. Both snapshot pictures are taken from animation1 and animation2.

can say that a given agent (here, the black square) belongs to the complete giant component if its nearest neighbour or next-nearest neighbour is an agent belonging to this component. The nearest neighbour case has jointed edges, while in the next-nearest neighbour case has vertices – both are visible in Fig. S1. The percolation cluster in the former case we call the autocatalytic percolation cluster.

We are interested in percolating clusters contained in the giant component, i.e., paths of agents connecting two opposite banks of the square lattice. The paths are also built/supported by agents who arrive at the giant component from outside, contributing to site percolation, while spin-offs can contribute to the autocatalytic bond percolation. Thus, it results in a mixed percolation-like cluster, where black squares are connected only by their edges (the red ramified/branched curve) and not by vertices, as the vertices' case occurs for a complete percolating cluster (see Fig. S1 for detail).

Passing through the multi-spectral criticality threshold, i.e., from the left to the right panels (or vice versa), can be treated as a symmetry-breaking phenomenon. We are dealing here with a structural phase transition of the order-disorder type.

Autocatalytic mixed percolation-like phase transition

The HP-phase structure resembles a canonical mixed-percolation cluster or ramified/branched chain, which we found using snap-shot pictures from MC simulations shown above. However, the form of the cluster observed dynamically changes which leads to fluctuating percolation interruptions. This is because of the following: (i) agents can change their places in the lattice by hopping between them, (ii) agents disappear due to mergers and agent death (bankruptcies), and (iii) agents repair a (fluctuating) percolating cluster by spin-offs of which these agents are parents. Furthermore, (iv) interactions between agents can disappear and reactivate. So the presence of agents in lattice sites is subject to local fluctuations. These are the fluctuations in the semi-stationary macrostate, which do not take the system out of semi-stationarity.

The triadic dependence/interaction/cooperation schematically shown in Fig. S2 (the left panel) is directly responsible for the construction of the fluctuating branched chain schematically presented in the right panel by circles and black links. Circles can disappear and reappear, while links can break and rejoin. We emphasise that interaction (marked by black links in the right panel) only occurs when the agents are their nearest neighbours. Notably, the next-nearest neighbours, e.g., agents in the pair (i, k), are marked by the dashed links. We limited our model to this assumption for simplicity because it does not change the essence of our considerations.

The other local configurations directly responsible for the dependence between agents are as follows (see Fig. S2 for detail): (i) two agents, one placed at the centre i and one at the north j, and a vacancy at the northeast or northwest as the nearest neighbour to the j and east or west to the i (e.g., $\hat{k}$ in Fig. S2 is the nearest neighbour to j), (ii) at the south nearest-neighbour position relative to center i, and (iii) similarly to the item (i), two agents placed at center & south, and a vacancy at the southeast or southwest position. Thus, the number of triads associated with a given pair of nearest-neighbour agents, (i, j), is $z_1 \cdot (z_1 - 1 + z_2)$, where $z_1 (= 4)$ is the number of nearest neighbours to the

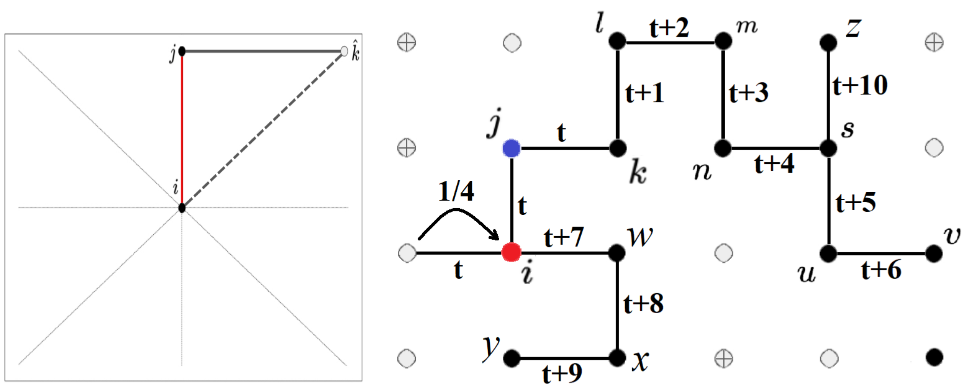

Fig. S2. Interacting agents (black and coloured circles) located in some square lattice sites. *Left panel*: Triadic plaquette. The zones of the nearest- and next-nearest neighbours of the occupied lattice site i (the black circle in the center) have the same number of $z_1 = z_2 = 4$ sites (as we deal with the square lattice). We assume an agent occupies the nearest-neighbour site j (also black circle). The pair of (i, j) are distinguished by a red bond, emphasising that they remain in the mutual dependence/interaction described in the main text in this subsection. The vacancy is located in site $\hat{k}$ (empty circle), belonging to the second coordination zone of site i. The black link means that the descendant placed in $\hat{k}$ can interact with the agent placed in j and not with the agent placed in i. Other triadic configurations we consider in the main text in this subsection. They have in common that the $\hat{k}$ site is the nearest- or next-nearest neighbour of the central site i. *Right panel*: The schematic example of a single realisation (in a single numerical experiment/simulation) of the ramified chain of the autocatalytic triads (i.e., a core of a giant component as shown in Fig. S1), having its origin at lattice site i at time t and its end at lattice site z at time $t + 10$. The '+' sign in the circle means a lonely agent in the position, not interacting with any neighbouring agent at a given MCS.

central location i (i.e., the size of the first coordination zone) and $z_2(=4)$ the number of next-nearest neighbours (i.e., the size of the second coordination zone). Of course, the number 4 in parentheses is characteristic only of the square lattice. As one can see, the spin-off can only be created in the nearest or next-nearest empty site, $\hat{k}$, to the central lattice site, i. The spin-off can interact, i.e., autocatalytic, with one of the parents in the nearest-neighbour lattice site.

For example, the percolating cluster shown in Fig. S2 (in the right panel) was formed by a sequence of the following autocatalytic reactions, except the first one, which initiated the reaction chain by producing a spin-off at the empty lattice site $\hat{k}$:

$$\begin{aligned} t:&\ A_i + A_j \to A_k, \\ t:&\ A_j + A_k \to A_l, \\ t+1:&\ A_k + A_l \to A_m, \\ t+2:&\ A_l + A_m \to A_n, \\ &\vdots \\ t+8:&\ A_w + A_x \to A_y, \end{aligned} \tag{S4}$$

where the catalyst efficiency, η , implicitly enters through the fitness value, defined in the main text. Let us note that the result of company activity in the subsequent moments $t+9$ and $t+10$ is not shown in Fig. S2.

A series of reactions defined by Eq. (S4) creates positive feedback in the considered system, underlying the percolating ramified/branched chain. The branching is because, for example, a pair of agents (n,s) can produce spin-offs in the step $t+4$ both in the position u and z, provided that these places were empty before. A similar situation occurs at t when spin-offs are created at k and w by a pair of (i,j) companies. It is also possible that a pair of companies will create a spin-off at a given moment, and the next spin-off will be created later.

The continuous phase transition discussed in this work is a structural phase transition, i.e., the transition from the (diluted or random) LP phase as introduced in the main text, where there is no structure and companies' activity is low, to the HP phase (LP and HP phases having been defined in the main text), which has the complex structure discussed above and high companies' activity.